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LETTER TO THE EDITOR

First integrals of autonomous systems of differential equations and the Prelle–Singer procedure

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Abstract. The Prelle–Singer procedure for determining elementary first integrals of two-dimensional autonomous systems of ordinary differential equations is introduced and how it can be generalized to higher dimensions is discussed. The application of this procedure in several dynamical systems is reported.

Given an autonomous system of first-order differential equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \omega_i(\boldsymbol{x}; \boldsymbol{a}) \qquad 1 \leqslant i \leqslant n \tag{1}$$

where t is the independent variable, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ is a set of dependent variables, $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$ is a set of parameters and ω_i are polynomials in $\boldsymbol{x}$, we are interested to know whether (1) has first integrals or not. By definition, a first integral is a non-constant function and its derivative (WRT t) vanishes on the solution curves of (1). So if $f(\boldsymbol{x})$ and $f(\boldsymbol{x})\mathrm{e}^{kt}$ are time-independent and time-dependent first integrals respectively, then we have

$$\sum_{i=1}^{n} \omega_i \frac{\partial f}{\partial x_i} = 0 \tag{2}$$

and

$$kf + \sum_{i=1}^{n} \omega_i \frac{\partial f}{\partial x_i} = 0. \tag{3}$$

If $f(\boldsymbol{x})$ is a polynomial, we usually call $f(\boldsymbol{x})\mathrm{e}^{kt}$ a quasi-polynomial first integral. One approach (see [5]) for determining polynomial or quasi-polynomial first integrals is to substitute the polynomial $f(\boldsymbol{x})$ with a given degree but undetermined coefficients into (2) or (3) and then solve the system of algebraic equations obtained by equating the coefficients of the left-hand sides of (2) or (3) to zero. How can we proceed if the system has first integrals which are not polynomials nor quasi-polynomials? For example, the predator–prey equations (see [4])

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bxy \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -cy + dxy \tag{4}$$

have an elementary first integral $dx + by - c\log x - a\log y$ (note: the special case when $a = b = c = d = 1$ was solved in [8] via the Carleman linearization technique) and we are interested to know how it can be found algorithmically. In 1983, Prelle and Singer proposed a procedure for determining elementary first integrals of two-dimensional

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autonomous systems of differential equations. Despite the fact that it is still a semi-decision procedure, it has been proved (e.g. see [2]) experimentally to be a very useful and practical procedure. It works as follows.

(i) Set $N = 1$.
(ii) Find all monic (i.e. the leading coefficient is unity) irreducible polynomials f_i with degrees $\leqslant N$ such that f_i divides Df_i, where D is the differential operator $\omega_1\partial/\partial x_1 + \omega_2\partial/\partial x_2$.
(iii) Let $Df_i = f_i g_i$. Decide if there are constants n_i, not all zero, such that

$$\sum_{i=1}^{m} n_i g_i = 0 .$$

If such n_i exist then $\prod_{i=1}^{m} f_i^{n_i}$ is a first integral. If no such n_i exist then go to the next step.
(iv) Decide if there are constants n_i, such that

$$\sum_{i=1}^{m} n_i g_i = -\left(\frac{\partial\omega_1}{\partial x_1} + \frac{\partial\omega_2}{\partial x_2}\right) .$$

If such n_i exist, then $R = \prod_{i=1}^{m} f_i^{n_i}$ is an integrating factor and we can obtain an elementary first integral I by integrating the following pair of equations:

$$\frac{\partial I}{\partial x_1} = R\omega_2 \qquad \text{and} \qquad \frac{\partial I}{\partial x_2} = -R\omega_1 .$$

If no such n_i exist, then go to the next step.
(v) Increase the value of N by 1. If N is greater than the preset bound then return failure, otherwise repeat the whole procedure.

Some remarks concerning this procedure are as follows.

(a) In most cases, the determination of the monic irreducible polynomials f_i for a given degree bound N is the most involved part of this procedure. Details of how f_i are computed can be found in [2].
(b) If we are interested in determining time-dependent first integrals then we need to modify the equation in step (iii) to $\sum_{i=1}^{m} n_i g_i = 1$. If such n_i exist, then $e^{-t}\prod_{i=1}^{m} f_i^{n_i}$ is such an integral. But we do not know yet what should be the corresponding step (iv) when step (iii) fails in such a case.
(c) The first integrals obtained in step (iii) are usually called *rational* first integrals (or *quasi-rational* first integrals if they are time-dependent) in order to distinguish from those integrals obtained in step (iv) via integration.

The generalization of this procedure to higher dimensions to find elementary first integrals is still an open problem. Nevertheless, we can modify the differential operator D in steps (ii) and (iii) to $\sum_{i=1}^{n} \omega_i\partial/\partial x_i$ to search for rational or quasi-rational first integrals for higher dimensional ($n \geqslant 3$) autonomous systems of differential equations. This procedure can be automated by means of computer-algebra systems and has been implemented in REDUCE by the author. All the examples mentioned in [5, 7] can be rediscovered by the implemented program. In addition, some further results can be obtained. For example, in the three-dimensional Lotka–Volterra model

$$\dot{x}_1 = x_1(1 + ax_2 + bx_3) \qquad \dot{x}_2 = x_2(1 - ax_1 + cx_3) \qquad \dot{x}_3 = x_3(1 - bx_1 - cx_2)$$

where a, b, c are real parameters (note: $\dot{x}_2$ was misprinted as $x_2(1 - ax_1 + bx_3)$ in [7] as was pointed out in [6]),

(i) if $a \neq 0$, $b = -2a$ and $c = 2a$, then up to fifth order, there exist two time-dependent first integrals

$$x_1^2 x_2^2 x_3 \mathrm{e}^{-5t} \qquad \text{and} \qquad (x_1 + x_2 + x_3)\mathrm{e}^{-t}$$

(ii) if $a \neq 0$, $b = -a$ and $c = 3a$, then up to fifth order, there exist two time-dependent first integrals

$$x_1^3 x_2 x_3 \mathrm{e}^{-5t} \qquad \text{and} \qquad (x_1 + x_2 + x_3)\mathrm{e}^{-t} .$$

We also discovered that the result (12) in [7] is actually dependent on two simpler results, namely $(x_1 + x_2 + x_3)\mathrm{e}^{-t}$ and $x_1 x_2 x_3 \mathrm{e}^{-3t}$. Similarly, we found that the result (13) in [7] is dependent on $(x_1 + x_2 + x_3)\mathrm{e}^{-t}$ and $x_1^2 x_2 x_3 \mathrm{e}^{-4t}$.

For the May–Leonard system (see [3])

$$\begin{aligned}
\dot{x_1} &= x_1(1 - x_1 - ax_2 - bx_3) \\
\dot{x_2} &= x_2(1 - bx_1 - x_2 - ax_3) \\
\dot{x_3} &= x_3(1 - ax_1 - bx_2 - x_3)
\end{aligned}$$

we discovered the following first integrals up to first order:

$$(x_1 + x_2 + x_3)^3 / x_1 x_2 x_3 \qquad \text{and} \qquad (x_1 + x_2 + x_3 - 1)\mathrm{e}^t / (x_1 + x_2 + x_3)$$

when $a + b = 2$.

Finally, it is interesting to point out that the algebraic invariant curves (AICs) mentioned in [1] are in fact the f_i polynomials in step 2 of the Prelle–Singer procedure. Hewitt showed that for restricted parameter values, a number of classes of cosmological models expressed in terms of polynomial systems of differential equations admit AICs, which, in turn, lead to exact solutions of the Einstein field equations. That means we can apply the Prelle–Singer procedure to automate the process of searching for AICs as well as finding the first integrals. For the Bianchi V model with a perfect fluid whose equation of state is $p = (\gamma - 1)\rho$ (see [1]):

$$\Sigma_-' = (q - 2)\Sigma_- \qquad A' = qA$$

with $q = 2\Sigma_-^2 + (3\gamma - 2)\Omega/2$ and $\Omega = 1 - A^2 - \Sigma_-^2$, we can obtain the following AICs:

$$A = 0 \qquad \Sigma_- = 0 \qquad 1 - A^2 - \Sigma_-^2 = 0$$

and the two first integrals

$$\Sigma_- \mathrm{e}^{2t} / A \qquad \text{and} \qquad \Omega A^{3(\gamma/2 - 1)} \Sigma_-^{1 - 3\gamma/2} .$$

What these integrals imply for the physically meaningful states of the cosmological model concerned can be left to the cosmologists to find out. But it is worth pointing out that the Prelle–Singer procedure can be applied to dynamical systems in different areas.

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